STUDENT NUMBER: ____________________

TEACHER: ____________________

BAULKHAM HILLS HIGH SCHOOL

YEAR 12

HALF YEARLY EXAMINATION

2010

MATHEMATICS
EXTENSION 1

GENERAL INSTRUCTIONS:

- Attempt **ALL** questions.
- Start each of the 7 questions on a new page.
- All necessary working should be shown.
- Write your teacher's name and your name on the cover sheet provided.
- At the end of the exam, staple your answers in order behind the cover sheet.
- Marks indicated for each question are only a guide and could change.

QUESTION 1 **Marks**

(a) Evaluate **1**

$$\lim_{x\to 0} \frac{sin2x}{3x}$$

(b) (i) Sketch the graph of $y = -x(x+2)(x-3)$ without using calculus. **2**

(ii) Solve $\frac{6}{x} > x - 1$ **3**

(c) If A and B are the points $(2, -1)$ and (-3,5) respectively, find the co- ordinates of the point $\mathrm{P}(x, y)$ that divides the interval AB externally in the ratio $3: 4$. **2**

(d) (i) Show that the curves $y = sin2x$ and $y = cos2x$ intersect at $x = \frac{\pi}{8}$. **1**

(ii) Find the acute angle between the two curves at $x = \frac{\pi}{8}$. **3**

QUESTION 2 **(Start a new page)**

(a) Find the exact value of $cos15°$ **2**

(b) If α, β and δ are the roots of the cubic $2x^3$+$6x^2 - 4x + 5 = 0$, find :

(i) $\alpha + \beta + \delta.$ **1**

(ii) $\alpha\beta + \alpha\delta + \beta\delta.$ **1**

(iii) α^2+ $\beta^2 + \delta^2.$ **2**

(c) Differentiate $log_e\sqrt{\frac{x+1}{x-1}}$. **2**

(d) (i) Express $sinx + \sqrt{3}cosx$ in the form $Asin(x + \alpha)$. **2**

(ii) Hence solve $sinx + \sqrt{3}cosx = 1$ for $0 \le x \le 2\pi$. **2**

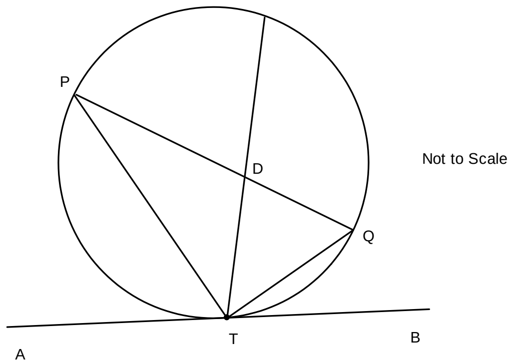
QUESTION 3 (Start a new page) **Marks**

(a) Solve $2\log_e(x+2) = \log_e(5x+6)$ **2**

(b) Evaluate $\int_0^{\frac{\pi}{3}} 2cos^2 x \ dx.$ **3**

(c) Taking $x = 2.5$ as the first estimate for the root of $f(x) = sinx - lnx$, use one application of Newton's method to find a better estimate for the root to 3 decimal places. **3**

(d) In the diagram below ATB is a tangent and PT and QT bisect the angles $\angle ATD$ and $\angle DTB$ respectively.



(i) Redraw this diagram on your page then prove that PQ is the diameter of the circle. **2**

(ii) Prove PQ ⊥ DT. **2**

QUESTION 4

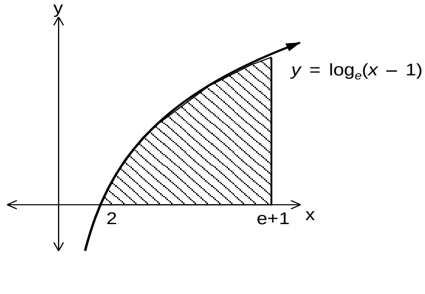
(a) The rate at which a body cools is proportional to the difference between the temperature (T) of the body and the surrounding temperature (C).

ie. $\frac{dT}{dt} = k(T - C)$

(i) Prove that $T = C + Ae^{kt}$ is a solution to the differential equation above. **1**

(ii) A heated body cools from $100°C$ to $60°C$ in an hour, after being placed in a room with a temperature of $20°C$. Find the temperature of the body after a further 2 hours. **3**

(b) Find the area of the shaded region below: **4**



(c) **(i)** Prove $2^2 + 2^3 + 2^4 + ... + 2^{n+1} = 2^2(2^n - 1)$ by mathematical induction. **3**

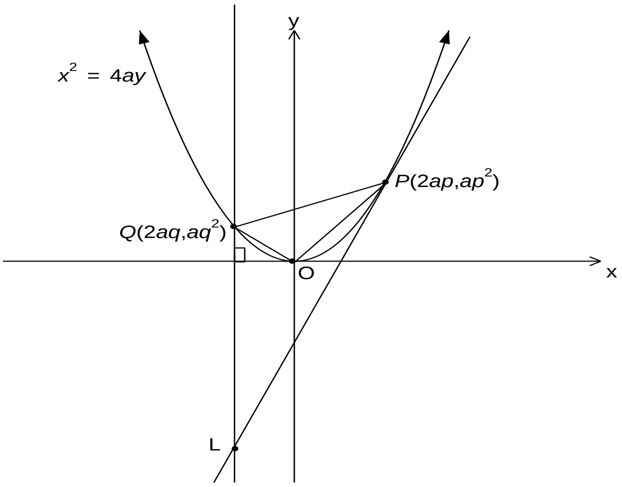
(ii) Hence evaluate $2^2 + 2^3 + ... + 2^{18}$. **1**

QUESTION 5 (Start a new page) **Marks**

(a) **(i)** Sketch on the same set of axes the graphs of $y = 2x + 1$ and $y = |x - 2|$ **2**

(ii) Hence or otherwise solve $|x - 2| < 2x + 1$ **2**

(b) The points P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.



(i) Find the gradient of OP. **1**

(ii) The chord PQ subtends a right angle at the origin. Show that $pq = -4$. **2**

(iii) Show that the equation of the tangent at P is $y = px - ap^2$. **2**

(iv) The tangent at P meets the line through Q perpendicular to the x axis at L. Show that L has co-ordinates $(2aq, 2apq - ap^2)$. **1**

(v) Find the locus of L. **2**

QUESTION 6 (Start a new page)

(a) Evaluate $\int_0^{\frac{2\pi}{3}} sec^2 x tan^2 x \; dx$ **3**

(b) If $\log_a 3 = x$ and $\log_a 4 = y$ express $\log_3 6$ in terms of x and y. **2**

(c) Hayden invests $2000 each year in a superannuation fund which earns 5% compound interest per annum for n years.

(i) How much does his first investment amount to after n years? **1**

(ii) Show that the total of his investments after n years is $42000(1.05^n - 1)$ **2**

(iii) Find the value of n if the total of his investments after n years is $93 454.20. **1**

(iv) $y = ax^3 - 7x^2 + bx + 20$ has a double root at $x = 2$. Find the values of a and b. **3**

QUESTION 7 **(Start a new page)** **Marks**

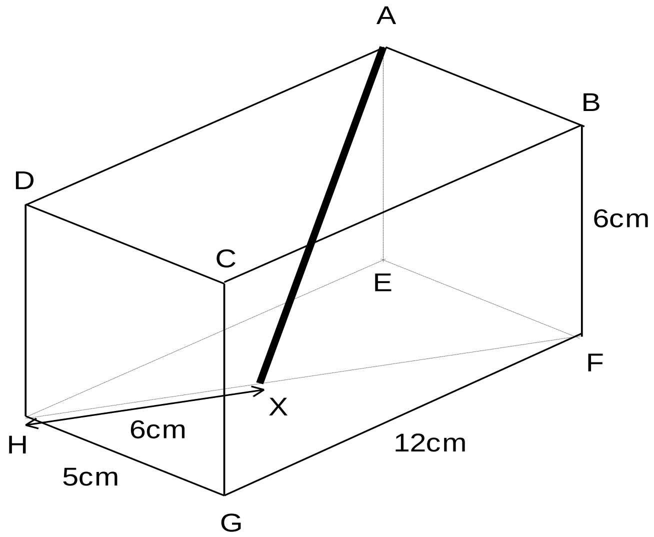
(a) Prove

$$\frac{sin^3 a + cos^3 a}{sin^2 a - cos^2 a} = \frac{cosec\, a + \cot a}{1 + \cot a}$$

3

(b) If $\frac{9^x + 6^x}{15^x + 10^x} = a^x$ find a. **2**

(c)



(i) An open pencil case in the shape of a rectangular prism has dimensions $5\ cm$ by $6\ cm$ by $12\ cm$. A pencil AX is placed in the case such that it rests at points A and X where X is $6\ cm$ along the diagonal FH. What angle does the pencil AX make with the base (plane EFGH)? **3**

(ii) A lid is placed on the pencil case. An ant stands at point D .It walks on the outside of the pencil case to F. What is the shortest distance from D to F? **1**

(d) **(i)** A function $y = f(x)$ has the following properties :

$$y' = \frac{1}{2}y$$
$$y'' = \frac{1}{2}y'$$
$$y''' = \frac{1}{2}y'' \quad \text{etc.}$$

Give a possible equation for $y = f(x)$. **1**

(ii) Find **2**

$$\lim_{n\to\infty} (y' + y'' + y''' + \cdots \quad + y^n)$$

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STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \ \ n \neq -1; \ \ x \neq 0, \ \ if \ \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \ \ a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \ \ a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \ \ a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \ \ a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \ a > 0, \ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \ \ x > a > 0$$

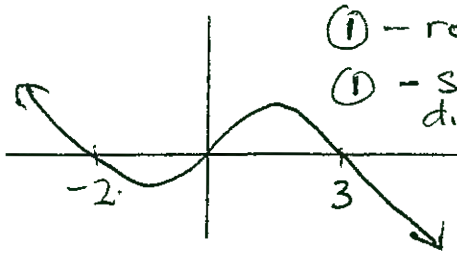
$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \ \ x > 0$

Solutions.

1 a) $\lim_{x\to 0} \frac{\sin 2x}{3x} = \frac{2}{3}$ (1)

b(i)



(1) – roots
(1) – shape + direction

(ii) $\frac{6}{x} > x-1$

$6x - x^2(x-1) > 0$ (1)

$x(6 - x(x-1)) > 0$

$x(6 - x^2 + x) > 0$

$-x(x^2 - x - 6) > 0$

$-x(x+2)(x-3) > 0$ } (1)

$x < -2$ or $0 < x < 3$ (1)

c) $(2,-1)$ $(-3,5)$

$-3 : 4$

(1) mark for $\left(\frac{-1}{7}, \frac{11}{7}\right)$

(1) (1)

$\left(\frac{-3(-3)+4(2)}{-3+4}, \frac{-3(5)+4(-1)}{-3+4}\right)$

$= (17, -19)$

d(i) $\sin 2x = \cos 2x$

$\tan 2x = 1$

$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \ldots$

$x = \frac{\pi}{8}, \ldots$ (1)

$m_1 = 2\cos 2\left(\frac{\pi}{8}\right)$ $m_2 = -2\sin 2\left(\frac{\pi}{8}\right)$

$m_1 = \frac{2}{\sqrt{2}}$ $m_2 = -\frac{2}{\sqrt{2}}$ (1)

$\tan\theta = \left|\frac{\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}}{1 + \frac{2}{\sqrt{2}}\cdot\frac{-2}{\sqrt{2}}}\right| = \left|\frac{\frac{4}{\sqrt{2}}}{-1}\right|$ (1)

$\theta = 70°32'$ (1)

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Question 2

a) $\cos 15° = \cos(45 - 30°)$

(1) $= \cos 45 \cos 30 + \sin 45 \sin 30$

$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ or $\frac{\sqrt{6}+\sqrt{2}}{4}$ (1)

b) $2x^3 + 6x^2 - 4x + 5 = 0$

(i) $\alpha+\beta+\gamma = -\frac{6}{2}$ (ii) $\alpha\beta+\alpha\gamma+\beta\gamma = -\frac{4}{2}$

(1) $= -3$ (1) $= -2$

(iii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$ (1)

$= (-3)^2 - 2(-2)$

$= 13$ (1)

c) $y = \log_e\sqrt{\frac{x+1}{x-1}}$

$= \frac{1}{2}\left[\log(x+1) - \log(x-1)\right]$ (1)

$\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x+1} - \frac{1}{x-1}\right]$

$= \frac{-1}{(x+1)(x-1)}$ } (1)

d(i) $A\sin x\cos\alpha + A\cos x\sin\alpha = \sin x + \sqrt{3}\cos x$

$A\cos\alpha = 1$ $A\sin\alpha = \sqrt{3}$

$\therefore A = \sqrt{(\sqrt{3})^2 + 1^2}$ $\tan\alpha = \sqrt{3}$

$= 2$ (1) $\alpha = \frac{\pi}{3}$ (1)

(ii) $2\sin\left(x + \frac{\pi}{3}\right) = 1$

$\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

$\therefore x + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$ (1)

$x = -\frac{\pi}{6}, \frac{\pi}{2}$

$\therefore x = \frac{11\pi}{6}, \frac{\pi}{2}$ (1)

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Question 3.

a) $2\log_e(x+2) = \log_e(5x+6)$

$\log_e(x+2)^2 = \log_e(5x+6)$

$\therefore (x+2)^2 = 5x+6$ (1)

$x^2+4x+4-5x-6=0$

$x^2-x-2=0$

$(x-2)(x+1)=0$

(1) $x=2, -1$ ($x>-2$ ✓)

b) $\int_0^{\frac{\pi}{3}} 2\cos^2 x \, dx$

$= \int_0^{\frac{\pi}{3}} (\cos 2x + 1)\, dx$ (1)

$= \left[\frac{1}{2}(\sin 2x) + x\right]_0^{\frac{\pi}{3}}$

$= \left(\frac{1}{2}\sin\left(\frac{2\pi}{3}\right) + \frac{\pi}{3}\right) - (0+0)$

$= \frac{\sqrt{3}}{4} + \frac{\pi}{3}$

c) $x_1 = 2.5 - \frac{f(2.5)}{f'(2.5)}$

$f(2.5) = \sin(2.5) - \ln(2.5)$

$= -0.3178\ldots$ (1)

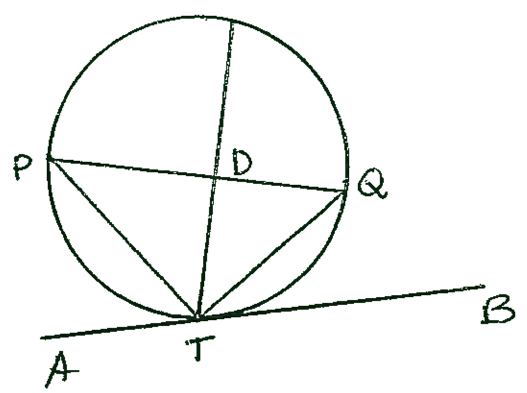
$f'(x) = \cos x - \frac{1}{x}$

$f'(2.5) = \cos 2.5 - \frac{1}{2.5} = -1.20\ldots$ (1)

$x_1 = 2.5 - \frac{-0.3178}{-1.20}$

$= 2.235$ (1)

d(i)



let $\angle ATP = x^\circ$

$\therefore \angle PTD = x^\circ$ (Given PT bisects $\angle ATD$)

let $\angle DTQ = y^\circ$

$\therefore \angle QTB = y^\circ$ (Given QT bisects $\angle DTB$)

$\therefore 2x+2y = 180^\circ$ (straight angle) (1)

hence $x+y = 90^\circ$

$\therefore \angle PTQ = 90^\circ$

If $\angle PTQ = 90^\circ$ then PQT must

(1) be in a semi circle

$\therefore$ PQ is the diameter.

(ii) $\angle ATP = \angle PQT = x^\circ$ (1)

(Angle between tangent & chord = angle in the alternate segment)

(1) $\therefore \angle TDQ = 180 - (x+y^\circ)$ (Angle sum of Δ)

$\therefore \angle TDQ = 90^\circ$

$\therefore$ DT $\perp$ PQ

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Question 4

a) (i) $T = C + Ae^{kt} \to T - C = Ae^{kt}$

$\frac{dT}{dt} = k(T-C)$

$\frac{dT}{dt} = kAe^{kt}$
$= k(T-C)$ (1)

$\therefore T = C + Ae^{kt}$ satisfies $\frac{dT}{dt} = k(T-C)$

(ii) $C = 20$

$T = 20 + Ae^{kt}$ when $t=0$ $T=100$

$\therefore 100 = 20 + Ae^{0}$

$A = 80$ (1)

$\therefore T = 20 + 80e^{kt}$

when $t=1$ $T=60$

$\therefore 60 = 20 + 80e^{k(1)}$

$\frac{1}{2} = e^{k}$

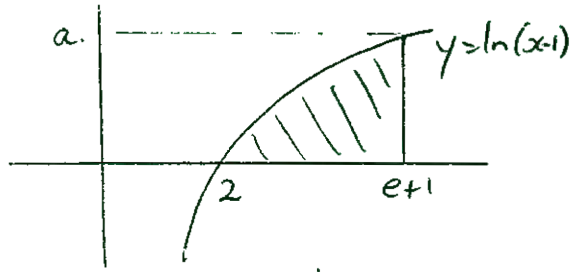
(1) $\ln\left(\frac{1}{2}\right) = k \ln e$
$k = -0.693\ldots$

when $t = 3$

$T = 20 + 80e^{-0.693..(3)}$

$T = 30$ (1)

b)



Area = 

$a = \ln(e+1-1)$
$= 1$

$\therefore$ Rectangle $= (e+1) \times 1$
$= e+1$ (1)

$y = \log_e(x-1) \therefore x - 1 = e^y$
$x = e^y + 1$

Area $= \int_0^1 (e^y + 1)\,dy$ (1)

$= \left[e^y + y\right]_0^1$ (1)

$= (e+1) - (e^0 + 0)$

$= e$

$\therefore$ Required area $= e + 1 - e$
$= 1$ (1)

c) Prove $2^2 + 2^3 + \ldots + 2^{n+1} = 2^2(2^n - 1)$

Step 1. Prove true for $n=1$

ie $2^2 = 2^2(2^1 - 1)$
$4 = 4$ ✓

Step 2 Assume true for $n=k$.

ie $2^2 + \ldots + 2^{k+1} = 2^2(2^k - 1)$ (1)

Step 3. Prove true for $n=k+1$

ie $2^2 + \ldots + 2^{k+1} + 2^{k+2} = 2^2(2^{k+1} - 1)$

by assumption need to prove

$\Rightarrow 2^2(2^k - 1) + 2^{k+2} = 2^2(2^{k+1} - 1)$ (1)

LHS $= 2^{k+2} - 2^2 + 2^{k+2}$

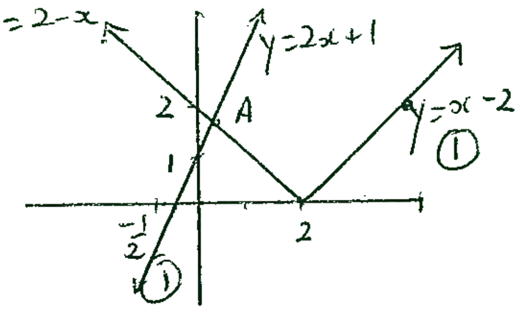
$= 2(2^{k+2}) - 2^2$
$= 2^{k+3} - 2^2$ (1)

$= 2^2(2^{k+1} - 1)$

$=$ RHS.

Step 4 Proved true for $n=1$ & assumed true for $n=k$ & prove true for $n=k+1$ $\therefore$ true for $n=1, n=2, \ldots$ & for all n by mathematical induction.

5a) (i)



(ii) Need A. solve simult.

$y = 2x + 1$ & $y = 2 - x$

$2x + 1 = 2 - x$

$3x = 1$

$x = \frac{1}{3}$ (1)

$\therefore$ solution to $|x-2| < 2x+1$

$x > \frac{1}{3}$ (1)

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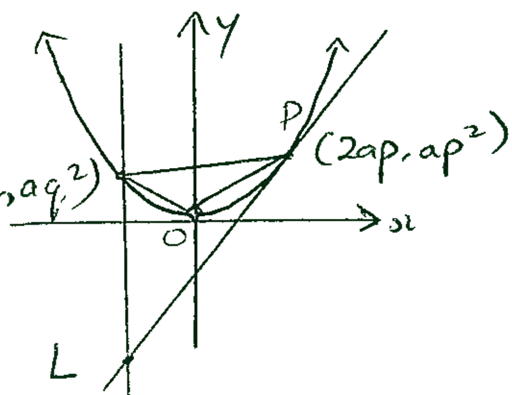
b)



(i) $m_{OP} = \frac{ap^2 - 0}{2ap - 0}$

$= \frac{p}{2}$ (1)

(ii) $m_{OQ} = \frac{q}{2}$ lines $\perp$ $\therefore$ (1)

$\frac{p}{2} \times \frac{q}{2} = -1$ (1)

$\therefore pq = -4$

(iii) $x^2 = 4ay \rightarrow y = \frac{x^2}{4a}$

$y' = \frac{2x}{4a}$

at $x = 2ap$ $y' = \frac{4ap}{4a}$

$= p$ (1)

$\therefore$ tangent $y - ap^2 = p(x - 2ap)$

$y = px - ap^2$ (1)

(iv) at L $x = 2aq$ $\therefore$

$y = p(2aq) - ap^2$ (1)

$= 2apq - ap^2$

$\therefore L\ (2aq, 2apq - ap^2)$

$pq = -4$

$\therefore q = \frac{-4}{p}$

$\therefore x = 2a\left(\frac{-4}{p}\right)$

$x = -\frac{8a}{p} \Rightarrow p = \frac{-8a}{x}$ (1)

$y = 2apq - ap^2$

$y = 2a(-4) - a\left(\frac{-8a}{x}\right)^2$

$y = -8a - \frac{64a^3}{x^2}$

$x^2y = -8ax^2 - 64a^3$

$8ax^2 + x^2y + 64a^3 = 0$ (1)

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Question 6

a) $\int_0^{\frac{2\pi}{3}} \sec^2 x \tan^2 x \, dx$

$\frac{d}{dx}(\tan^3 x) = 3\sec^2 x \tan^2 x$ (1)

$\therefore \int_0^{\frac{2\pi}{3}} \sec^2 x \tan^2 x = \frac{1}{3}\left(\tan^3 x\right)_0^{\frac{2\pi}{3}}$ (1)

$= \frac{1}{3}\left[\tan^3 \frac{2\pi}{3} - \tan^3 0\right]$

$= \frac{1}{3} \cdot -(3\sqrt{3})$ (1)

$= -\sqrt{3}$.

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b) $\log_a 3 = x$ $\log_a 4 = y$

$\log_3 6 = \frac{\log_a 6}{\log_a 3}$ (1)

$= \frac{\log 3 \times \sqrt{4}}{\log_a 3}$

b) $= \dfrac{\log_a 3 + \frac{1}{2}\log_a 4}{\log_a 3}$

$= \dfrac{x + \frac{1}{2}y}{x}$

$= 1 + \dfrac{y}{2x}$ } (1)

c) (i) 2000×1.05^n (1)

(ii) Investments =

$2000 \times 1.05^n + 2000 \times 1.05^{n-1} + \dots$

$\dots + 2000 \times 1.05.$ } (1)

$S_n = 2000 \times 1.05 \left(\dfrac{1.05^n - 1}{1.05 - 1} \right)$

$= 42000(1.05^n - 1)$ (1)

(iii) $93454.20 = 42000(1.05^n - 1)$

$1.05^n = \dfrac{93454.2}{42000} + 1.$

$1.05^n = 3.2251$

$n = \dfrac{\ln(3.2251)}{\ln(1.05)}$

$n = 24$ } (1)

d) $y = ax^3 - 7x^2 + bx + 20$

(2,0) satisfies.

$0 = a(2)^3 - 7(2)^2 + 2b + 20$

$0 = 8a - 28 + 2b + 20$

$8a + 2b = 8$

$4a + b = 4$ --- (A) (1)

$y' = 3ax^2 - 14x + b$

when $x = 2$ $y' = 0$

$\therefore 0 = 3a(2)^2 - 14(2) + b$

$0 = 12a - 28 + b$ (1)

$12a + b = 28$ -- (B)

$4a + b = 4$ -- (A)

(B) − (A) $8a = 24$

$a = 3$

$b = -8$ } (1)

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Question 7.

a) $\dfrac{\sin^3 a + \cos^3 a}{\sin^2 a - \cos^2 a} = \dfrac{\operatorname{cosec} a \mp \cot a}{1 \mp \cot a}$

LHS.

$\dfrac{(\sin a + \cos a)(\sin^2 a - \sin a \cos a + \cos^2 a)}{(\sin a + \cos a)(\sin a - \cos a)}$ (1)

$\dfrac{1 - \sin a \cos a}{\sin a - \cos a}$ $\sin^2 a + \cos^2 a = 1$ (1)

$= \dfrac{\dfrac{1}{\sin a} - \dfrac{\sin a \cos a}{\sin a}}{\dfrac{\sin a}{\sin a} - \dfrac{\cos a}{\sin a}}$ (1)

$= \dfrac{\operatorname{cosec} a - \cos a}{1 - \cot a}$

= RHS.

b(i) $\dfrac{9^x + 6^x}{15^x + 10^x} = a^x$

LHS. $\dfrac{3^x(3^x + 2^x)}{5^x(3^x + 2^x)}$ ①

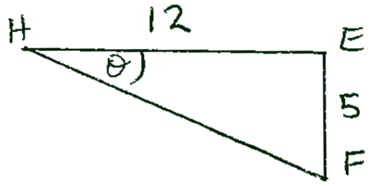
$= \left(\dfrac{3}{5}\right)^x$ ①

$a = \dfrac{3}{5}$

Bald = 0 $\dfrac{3\cdot 3^x + 3\cdot 2^x}{5\cdot 3^x + 5\cdot 2^x}$

Let $x =$ $a = \frac{3}{5}$ ✓

c) (i)



$\tan\theta = \dfrac{5}{12}$ 7°23'

$\theta = 22°37'$ ①

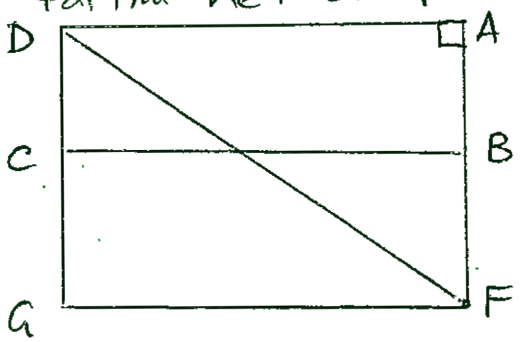
$EX^2 = 6^2 + 12^2 - 2.6.12.\cos 22°37'$

$EX = 6{\cdot}861$ ①

$\tan\angle AXE = \dfrac{6}{6.861}$

$\therefore \angle AXE = 41°10'$ ①

(ii) Partial net of prism



Short distance using Pythagoras

$DF = \sqrt{12^2 + 11^2}$

$= \sqrt{265}$ ①

d (i) $y = e^{\frac{1}{2}x}$ ①

(ii) $y' = \frac{1}{2} e^{\frac{x}{2}}$

$y'' = \frac{1}{4} e^{\frac{x}{2}}$

$y''' = \frac{1}{8} e^{\frac{x}{2}}$ etc ①

$\therefore \lim_{n\to\infty} \left(\frac{1}{2}e^{\frac{x}{2}} + \frac{1}{4}e^{\frac{x}{2}} + \dots\dots\right)$

$= e^{\frac{x}{2}}\left[\dfrac{\frac{1}{2}}{1-\frac{1}{2}}\right]$

$= e^{\frac{x}{2}}\,.\,1$

$= e^{\frac{x}{2}}$

$= y.$ ①

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trivial case

$\frac{1}{2}y$ $y = 0$ .... most an

$\frac{1}{2}y + \frac{1}{2}\times(\frac{1}{2}y) + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}y$ ----